Parametric Dictionary Design for Sparse Coding

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SPARS09, April 7, 2009

Outline

Sparse Coding

Generative model

$$\mathbf{y} = \mathbf{D}\mathbf{x} + \nu$$

$\mathbf{y} \in \mathbb{R}^d$, $\mathbf{D} \in \mathbb{R}^{d \times N}$, $\mathbf{x} \in \mathbb{R}^N$ and $\nu \in \mathbb{R}^d$. *Under-determined generative model*

$$\Leftrightarrow d < N$$

$$\overbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_d \end{bmatrix}}^{\mathbf{y}} = \overbrace{\begin{bmatrix} d_{1,1} & & d_{1,k} & & d_{1,N} \\ d_{2,1} & & d_{2,k} & & d_{2,N} \\ \vdots & \dots & \vdots & \dots & \vdots \\ d_{d,1} & & d_{d,k} & & d_{d,N} \end{bmatrix}}^{\mathbf{D}} \overbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \\ x_N \end{bmatrix}}^{\mathbf{x}} + \overbrace{\begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_d \end{bmatrix}}^{\nu}$$

- sparse coding: $\widehat{\mathbf{x}} = \arg\min_x \|\mathbf{x}\|_0 \text{ s.t. } \|\mathbf{y} - \mathbf{D}\mathbf{x}\|^2 \leq \xi,$
- $\xi = 0$ called sparse representation.
- $\xi > 0$ called sparse approximation.

Dictionary Selection Methods

- *Concatenation of orthonormal bases:* Let $\mathcal{O}$ be the set of all orthonormal dictionaries in $\mathbb{R}^{d\times d}$. $\mathcal{D} = \{\mathbf{D}_i\}_{i\in\mathcal{I}}, \forall i \in \mathcal{I}, \mathbf{D}_i \in \mathcal{O}$ is given. A dictionary $\mathbf{D}$ in $\mathbb{R}^{d\times d|\mathcal{I}|}$ is generated using,

$$\mathbf{D} = [\mathbf{D}_1 \cdots \mathbf{D}_i \cdots \mathbf{D}_{|\mathcal{I}|}].$$

- *Dictionary learning using a set of training samples $\mathscr{L} = \{\mathbf{y}_i\}_{i\in\mathcal{I}}$:* The goal is to find a dictionary such that it provides sparser coding for the given class of signals. A general formulation for dictionary learning is,

$$\widehat{\mathbf{D}} = \arg\min_{\mathbf{D}} \sum_{i\in\mathcal{I}} \min_{\mathbf{x}\in\{\theta:\|\mathbf{y}_i-\mathbf{D}\theta\|^2\leq\xi\}} \|\mathbf{x}\|_0.$$

- *Dictionary design subject to a certain property:* These properties include, but not restrict to, Restricted Isometry Property (RIP), minimum coherence μ and minimum cumulative coherence $\mu_1(m)$.

Incoherent Dictionary

Coherence

The coherence of a dictionary $\mathbf{D}$ is defined to be the maximum correlation of two distinct atoms and can be found using,

$$\mu_{\mathbf{D}} = \max_{i,j:j\neq i}\{|\langle \mathbf{d}_i, \mathbf{d}_j\rangle|\}.$$

Incoherent dictionary: A dictionary is incoherent when $\mu_{\mathbf{D}}$ is "small".

Equiangular Tight Frame (ETF)

A column normalized dictionary $\mathbf{D}_G$ is called ETF, when there is a $\gamma \ : \ 0 < \gamma < \pi/2$, such that $|\langle \mathbf{d}_i, \mathbf{d}_j\rangle| = \cos(\gamma) \ : \ \forall i, j \ i \neq j$.
If there exists an ETF in $\mathbb{R}^{d\times N}$, it is the solution of $\arg\min_{\mathbf{D}\in\mathscr{D}}\{\mu_{\mathbf{D}}\}$, and for any $\mathbf{D} \in \mathbb{R}^{d\times N}$, $\mu_{\mathbf{D}} \geq \mu_G := \sqrt{\frac{N-d}{d(N-1)}}$.

Parametric Dictionary

Parametric Dictionary

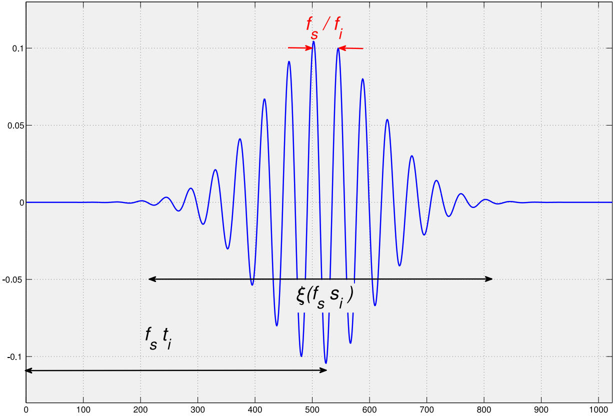
Let Γ be a set of parameters in the admissible set Υ.
$\mathbf{D}_\Gamma : \Upsilon \rightarrow \mathbb{R}^{d \times N}$ is defined to be a mapping from the parameter space to the dictionary space.
Given a mapping $\mathbf{D}_\Gamma$, a dictionary called Parametric Dictionary, when it is generated using a $\Gamma \in \Upsilon$.

Special Case: A special mapping $\mathbf{D}_\Gamma$ is when it is column-separable. In this framework each atom $\mathbf{d}_i$ is generated by a set of parameters γ_i. This type of mapping has often been used in sparse coding.

An example: Multi-scale Gabor Parametric Dictionary

The parameters in a real value multi-scale Gabor atom are scale, phase and the time-frequency shift and the generative function is Gaussian.

$$\mathbf{g}_{\gamma_i}(t) = a_i e^{\left(\frac{t-t_i}{s_i}\right)^2} . \cos(2\pi f_i (t - t_i) + \phi_i)$$



Parametric Dictionary Design

Parametric Dictionary Design

Let $\mathbf{D}_\Gamma$, $\boldsymbol{\Upsilon}$ and a certain dictionary property be given. Parametric Dictionary Design is how to find a $\Gamma^* \in \Upsilon$ such that $\mathbf{D}_{\Gamma^*}$ (approximately) has such a property.

- *Objective:* Finding a dictionary close to being ETF.
- This search is easier to be done in the space of N by N real matrices, by finding the Gram matrix, $\mathbf{G}_\mathbf{D} := \mathbf{D}^T\mathbf{D}$, associated with the optimal parametric dictionary.
- *Closeness measure:* The infinity norm $\|.\|_\infty$ in $\mathbb{R}^{N\times N}$, which is the maximum absolute value of the elements.

Parametric Dictionary Design: Formulation

- Let Θ_d^N be the set of Gram matrices of the ETF's in $\mathbb{R}^{d\times N}$. An incoherent PDD is found using the following optimization problem,

$$\arg \min_{\Gamma\in\Upsilon, \mathbf{G}_G\in\Theta_d^N} \|\mathbf{D}_\Gamma^T \mathbf{D}_\Gamma - \mathbf{G}_G\|_\infty$$

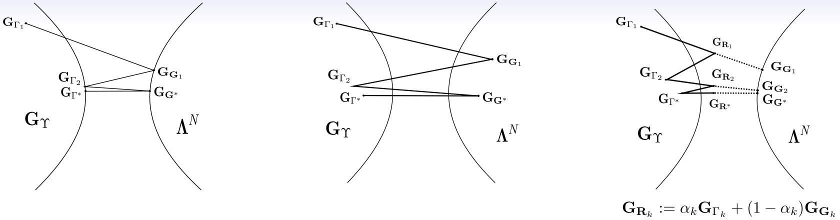
- *Difficulties:* There is no ETF for some selection of (d, N), the set of ETF's in $\mathbb{R}^{d\times N}$ is non-convex and it is difficult to minimize the ℓ_∞ problem.
- *Practical Solution:* Convex relaxation of the set of ETF's,

$$\Lambda^N = \{\mathbf{G}\in\mathbb{R}^{N\times N} : \mathbf{G}=\mathbf{G}^T, \operatorname{diag}\mathbf{G}=1, \max_{i\neq j}|g_{i,j}|\leq \mu_G\},$$

and using ℓ_2-norm as the closeness measure,

$$\arg \min_{\Gamma\in\Upsilon, \mathbf{G}\in\Lambda^N} \|\mathbf{D}_\Gamma^T \mathbf{D}_\Gamma - \mathbf{G}\|_F^2.$$

Optimization Methods



(a) (b) (c)

$$\min_{\mathbf{G}_\Gamma \in \mathbf{G}_\Upsilon, \mathbf{G}_\mathbf{G} \in \Lambda^N} \|\mathbf{G}_\Gamma - \mathbf{G}_\mathbf{G}\|_F^2$$

$$\mathbf{G}_\Upsilon = \{\mathbf{G}_\Gamma : \mathbf{G}_\Gamma = \mathbf{D}_\Gamma^T \mathbf{D}_\Gamma, \Gamma \in \Upsilon\}$$

(a) Alternating Projection: Needs to know the projection operators.

(b) Alternating Minimization: Reduces the objective at each parameter update.

(c) Relaxed Alternating Minimization : When the solution in one of the sets, here Υ, is needed to be found.

PDD Algorithm

Parametric Dictionary Design

1: **initialization:** $k = 1$, $\mathbf{D}_{\Gamma_1} \in \mathscr{D}$, $\{\alpha_i\}_{1 \le i \le K} : 0 < \alpha_i \le 1$
2: **while** $k \le K$ **do**
3: $\mathbf{G}_{\Gamma_k} = \mathbf{D}_{\Gamma_k}^T \mathbf{D}_{\Gamma_k}$
4: $\mathbf{G}_{P_{k+1}} = \arg\min_{\mathbf{G} \in \Lambda^N} \|\mathbf{G}_{\Gamma_k} - \mathbf{G}\|_F$
5: $\mathbf{G}_{R_{k+1}} = \alpha_k \mathbf{G}_{P_{k+1}} + (1 - \alpha_k) \mathbf{G}_{\Gamma_k}$
6: $\mathbf{D}_{\Gamma_{k+1}} \in \mathbf{D}_{\Gamma_k} \cup \{\forall \mathbf{D} \in \mathscr{D} : \|\mathbf{D}^T\mathbf{D} - \mathbf{G}_{R_{k+1}}\|_F < \|\mathbf{G}_{\Gamma_k} - \mathbf{G}_{R_{k+1}}\|_F\}$
7: $k = k + 1$
8: **end while**

4: *Projection of* $\mathbf{G}_{\Gamma_k}$ *onto* Λ^N*:*

$$\mathcal{P}_{\Lambda^N}(\mathbf{G}_\mathbf{D} = \mathbf{D}^T\mathbf{D}) = \{g_{P\,i,j}\}$$

$$g_{P\,i,j} = \begin{cases} \mathrm{sign}(g_{\mathbf{D}\,i,j})\mu_G & i \neq j \\ 1 & \textit{otherwise} \end{cases},$$

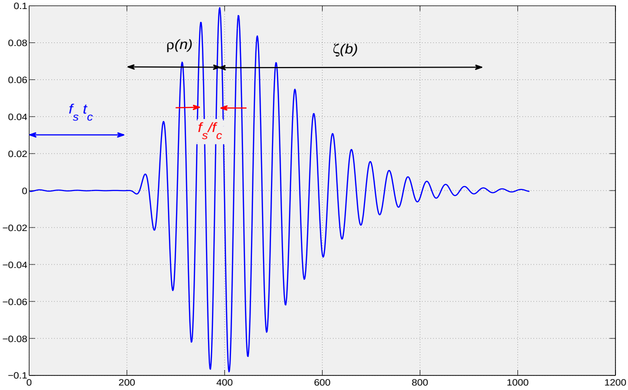
6: *Parameter update* $\Gamma_k \to \Gamma_{k+1}$*:* using a gradient descent method.

Case Study: Gammatone Parametric Dictionary

Gammatone Atom:

$$g(t-t_c) = a.(t-t_c)^{n-1} e^{-2\pi bB(t-t_c)} \cos(2\pi f_c(t-t_c))$$

$B = f_c/Q + b_{min}$, where Q is a constant and b_{min} is the minimum bandwidth.
$\gamma = [t_c \ f_c \ n \ b]^T$ is the set of parameters which could be found using PDD.



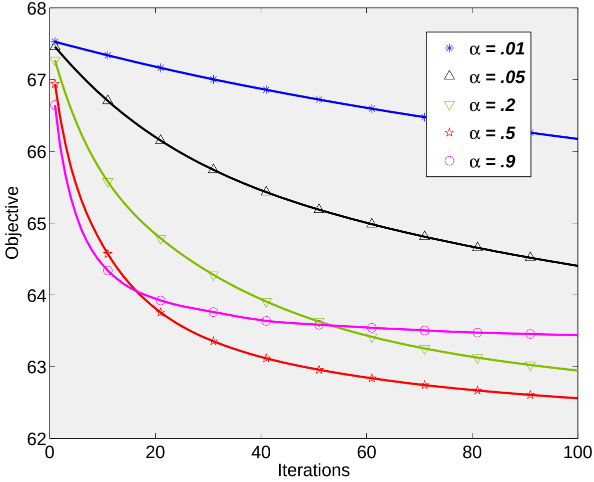
Simulation setting

General Parameters (ETF is plausible with this setting)	
Name	Value
d	1024
N	2048
Q	9.26449
b_{min}	24.7 Hz

Initial Parameters	
Name	Value
f_s	12000 Hz
n	4
b	0.5
f_c^k	$(f_s/2 + Qb_{min})e^{-0.45k/Q} - Qb_{min}$
t_c^l	$t_p + 0.75(l-1)\ t_p$

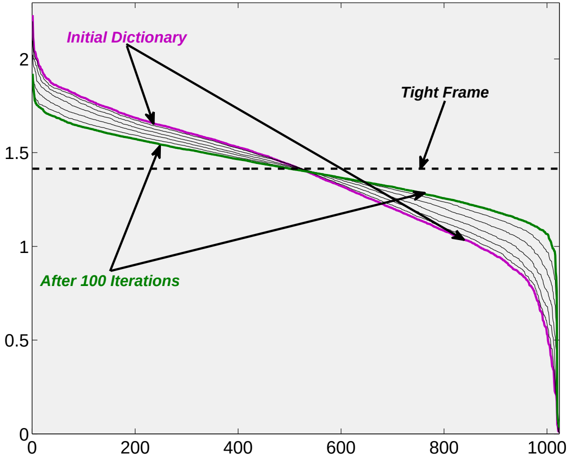
Simulations: Role of α

The objective functions for different $\{\alpha_k\}_{\forall k, \alpha_k = \alpha}$, for a constant α.



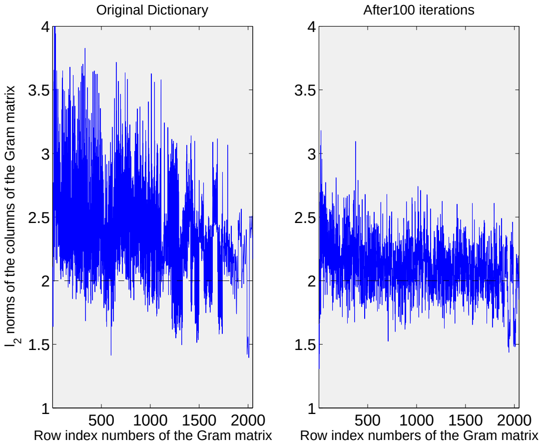
Simulations: Getting Close to Being Tight Frame

Singular values of the dictionaries at some selected iterations.



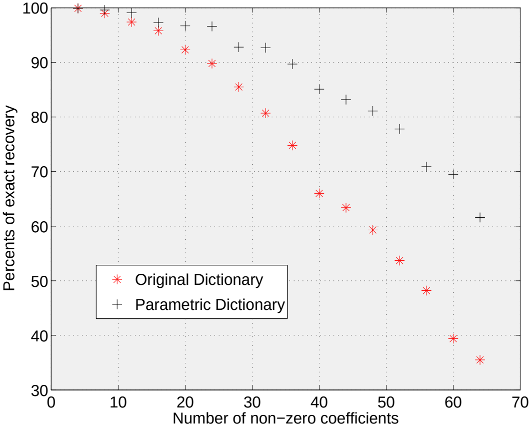
Simulations: Getting Close to Being ETF

The column ℓ_2-norm plots of the Gram matrices $(\sum_j |\langle \mathbf{d}_i, \mathbf{d}_j \rangle|^2)$ of the original (left) and designed (right) dictionaries. The column ℓ_2-norm of $\mathbf{G} \in \Lambda^N$ $(1 + (N-1)\cos^2(\gamma))$ is plotted for reference.



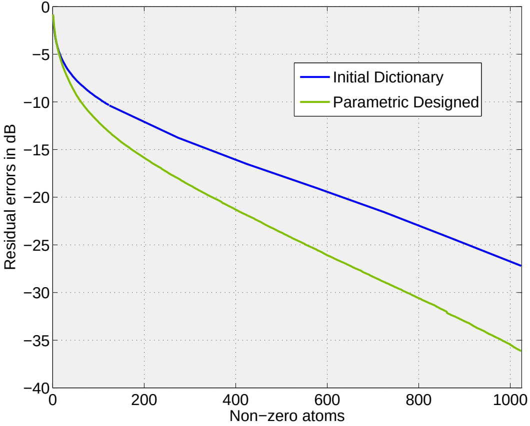
Simulations: Exact Recovery

Exact support recovery of the sparse signals.



Simulations: Residual Error Decay Rate of MP

The residual error plots using matching pursuit for sparse approximation of the audio signal.



Conclusion and Future Work

Conclusion

- A new framework for dictionary design, under a minimum coherence constraint, was presented .
- A difficult optimization problem needs to be solved $\Rightarrow$ a relaxed optimization problem was introduced.
- A practical algorithm was presented to *approximately* solve the relaxed problem.
- Simulations demonstrate that the proposed algorithm not only finds a dictionary which is close to being ETF, but also this dictionary practically shows superior performance in a real application.

Future Work

- Structured Parametric Dictionary Design, to find a fast dictionary.
- Finding a more efficient method for the parameter update step.
- Alternate design criterion.

Thanks for your attention.

Any question ?