Analysis Operator Learning for Overcomplete Co-sparse Representations

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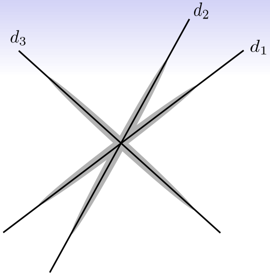
Outline

Analysis Framework : An Introduction

- A low dimensional signal model.
- A special type of the union of subspaces signal model.
- Has many applications in, for example, denoising, compressed sensing and inverse problems to improve the overall performance.



Analysis Model

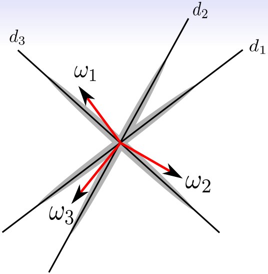
The signal $\mathbf{y}$ follows the model, if there exists a (linear) analysis operator $\Omega \in \mathbb{R}^{n \times m}$, $n \geq m$ that sparsifies $\mathbf{y}$,

$$\mathbf{z} = \Omega \mathbf{y}.$$

$\|\mathbf{z}\|_0 = n - p$, where $p > 0$ is called the co-sparsity of $\mathbf{y}$.

Analysis Operator Learning (AOL) Formulation

- A set of samples $\mathbf{Y} = [\mathbf{y}_1 \ \ldots \ \mathbf{y}_i \ \ldots \ \mathbf{y}_L]$ is given.
- The goal is to find a suitable analysis operator Ω such that $\|\Omega\mathbf{Y}\|_0$ is small.
- The objective is non-smooth $\Rightarrow$ not suitable for optimization with variational techniques.
- A relaxation is to select the sum of absolute values operator, *i.e.* $\|\cdot\|_1 = \sum_{ij} |\{\cdot\}_{ij}|$.



Formulation

The learned operator is found by minimizing the sparsity promoting operator,

$$\min_{\Omega} \|\Omega\mathbf{Y}\|_1 \ \text{ s. t. } \ \Omega \in \mathcal{C}$$

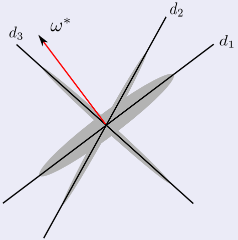
where $\mathcal{C}$ is a constraint, to exclude the trivial solutions, *e.g.* $\Omega = \mathbf{0}$.

Insufficient Constraints

Row norm constraints

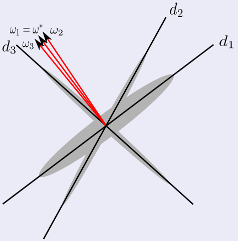
$$\forall i,\ \|\omega_i\|_2 = c$$

Rank one Ω_1 is found by repeating the best (almost) orthogonal direction ω^* to columns of $\mathbf{Y}$.



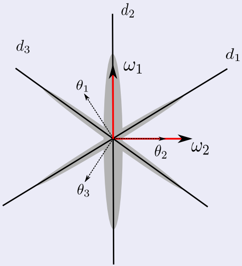
Row norm + full rank constraints

A randomly perturbed Ω from Ω_1 , *i.e.* row normalized $\Omega_1 + \mathbf{N}$, has a full rank and it is still not suitable.



Tight frame constraints

It resolves the issue in a complete setting. In the overcomplete cases, it includes zero-padded orthobases.



Proposed Constraint

Uniform Normalized Tight Frame (UNTF):

Definition: $\mathcal{C} = \{\Omega \in \mathbb{R}^{n \times m} : \Omega^T \Omega = \mathbf{I} \ \& \ \forall i \ \|\omega_i\|_2 = \sqrt{\frac{m}{n}}\}$

Pros and Cons:

- Zero-padded orthobases are not UNTF.
- Efficient methods exist to project onto the TF and the UN manifolds. However, there is no analytical way to find the projection onto the UNTF!
- There is no easy way to find the global optimum, using $\mathcal{C}$ as the constraint.

Local Identifiability

Definition

Let an analysis operator Ω_0 exist that the set of given training samples $\mathbf{Y}$ are cosparse. It is called "locally identifiable", if it is a local optimum of the proposed optimization problem.

- An admissible point Ω_0 is a local minimum of $\|\Omega\mathbf{Y}\|_1$, if any perturbation of Ω_0 in the tangent space of UNTF, increases the objective.
- We can then show the local optimality of Ω_0 by showing $\Delta = 0$ is the only solution of,

$$\min_{\Delta} \|(\Omega_0 + \Delta)\mathbf{Y}\|_1 \ \text{s. t.} \quad \Delta^T\Omega_0 + \Omega_0^T\Delta = 0$$

$$\forall i \ \ \langle \omega_{0i}, \delta_i \rangle = 0.$$

Projected Subgradient Algorithm for AOL

Motivation

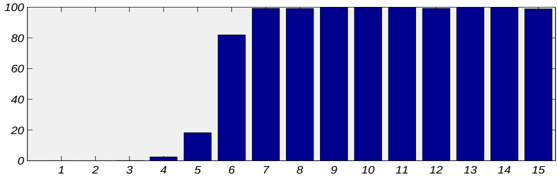
Minimization of a convex objective subject to the intersection of two easy to project admissible sets $\Rightarrow$ a variant of projected subgradient algorithm can be a good candidate.

Projected Subgradient Algorithm for AOL

initialization: $k = 1$, K_{max}, $\Omega^{[0]} = \mathbf{0}$, $\Omega^{[1]} = \Omega_{in}$, γ, $\epsilon \ll 1$
while $\epsilon \leq \|\Omega^{[k]} - \Omega^{[k-1]}\|_F$ and $k \leq K_{max}$ **do**
  $\Omega_G = \partial f(\Omega^{[k]})$
  $\Omega^{[k+1]} = \mathcal{P}_{UN}\left\{\mathcal{P}_{TF}\left\{\Omega^{[k]} - \gamma\Omega_G\right\}\right\}$
  $k = k + 1$
end while
output: $\Omega_{out} = \Omega^{[k-1]}$.

Exact Operator Recovery

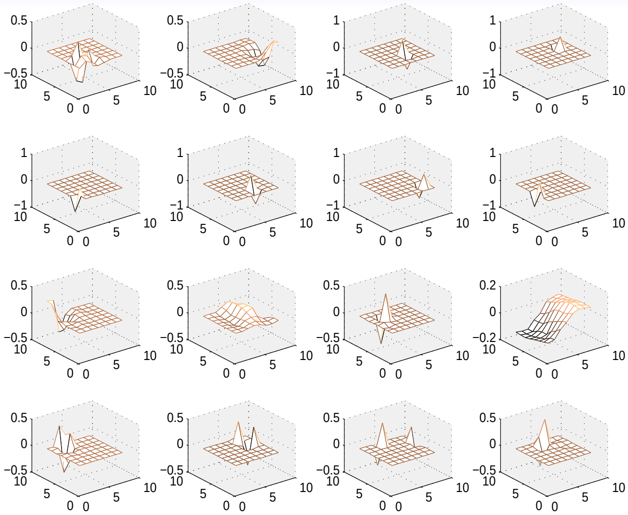
- A pseudo-random UNTF operator $\Omega_0 \in \mathbb{R}^{24\times 16}$ was used to generate $N = 768$ training samples.
- For each cosparsity p, a random normal vector was selected in the orthogonal complement space of p randomly selected rows of Ω_0.
- The simulation was started with a different pseudo-random admissible Ω_{in} and iterated 50000 times.
- The average recovery of the rows of Ω_0, for different cosparsities and 100 trials, is shown as a function of the cosparsity of the signals.



AOL for the Piecewise Constant Images

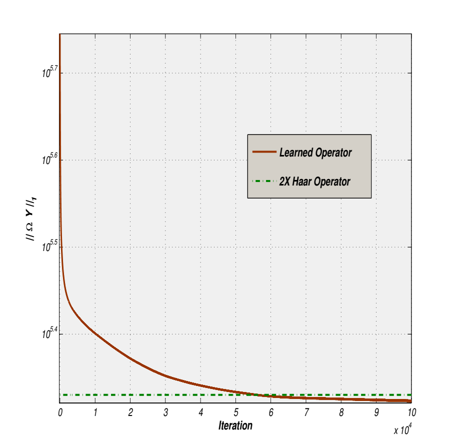
- Finding an Ω for the image patches of size 8×8.
- A 512×512 Shepp-Logan phantom image was used as the training image.
- $N = 16384$ image patches was randomly chosen from the training image.
- A pseudo-random UNTF operator $\Omega_0 \in \mathbb{R}^{128 \times 64}$ was used as the initial operator and the algorithm iterated 100,000 times!

AOL for the Piecewise Constant Images: the First 16 Learned Rows



A Comparison with Another UNTF

- Some rows have similarities with the finite difference operator rows $\longrightarrow$ finite difference operator is not a UNTF.
- An alternative is to use (orthonormal) Haar wavelet as the mother basis to generate a union of orthobases.
- The union of a Haar wavelet and a circularly shifted version, was selected for comparison.



Do we get any better operator by initializing with the generated Haar based operator? NO. It is indeed a local minimum for the proposed AOL program.

- The proposed analysis operator learning technique showed promising results in the exact operator recovery.
- Although the proposed constraint may not be the most relevant constraint, it works well with the piecewise constant images.
- Although each iteration of the AOL algorithm is not computationally expensive, it converges very slow.

Future Work

- Alternative constraints.
- Better optimization techniques.
- Deriving an explicit formulation for the recovery of an operator.

Thanks for your attention.

M. Yaghoobi, S. Nam, R. Gribonval, M. Davies, "Analysis Operator Learning for Overcomplete Cosparse Representations", European Signal Processing Conference (EUSIPCO), August, 2011.