Non-Negative Orthogonal Matching Pursuit

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***Abstract*—One of important classes of sparse signals is the non-negative sparse signals. Canonical greedy techniques have been modified to incorporate the non-negativity of the representations. One of such modification has been proposed for Orthogonal Matching Pursuit (OMP), where chooses positive coefficients first and uses a non-negative least square as a replacement for the orthogonal projection onto the selected support, at each iteration of the algorithm, which is computationally expensive. We here present a very different modification to the canonical OMP implementation, which truly incorporate the non-negativity of the coefficients. We also present a novel fast implementation of the Non-Negative OMP (NNOMP) which is based on the QR decomposition. As a result we show that we may receive an acceleration of a factor of up to ten in a reasonable size problem with the new method[1].**

I. Introduction

Let $\mathbf{y} \in \mathbb{R}^m$ be a signal having the sparse representation using a normalised linear generative model $\mathbf{\Phi} \in \mathbb{R}^{m\times n}$, *i.e.* dictionary, where $\mathbf{y} = \mathbf{\Phi x}$ and $\mathbf{x}$ is a sparse vector. Orthogonal Matching Pursuit gradually adds one element of the dictionary, called an atom, to the selected non-zero set and find the best possible representation of $\mathbf{y}$ using selected atoms, *i.e.* orthogonal projection. When the signal of interest has a sparse and positive representation, *i.e.* $\|\mathbf{x}\|_0 \leq k$, $\mathbf{x} \in \mathbb{R}^n_+$, we like to incorporate the extra information and adapt OMP to the new setting. This adaptation has been proposed in [1] with two modifications: a) only selecting the atoms with positive correlation with the residual of the signal in that iteration, *i.e.* $i^* = \text{argmin}_i \boldsymbol{\phi}_i^T \mathbf{r}^{[t]}$ where $\mathbf{r}^{[t-1]}$ is the orthogonal component of $\mathbf{y}$ to the span of currently selected atoms, and $\boldsymbol{\phi}_i$ is the ith atom, b) using Non-Negative Least Square (NNLS) representation within selected atoms. The latter step is indeed computationally very expensive for large k's. As we have a non-negativity constraint on the representation, we may find that the selected atom in each iteration forces the coefficients of some already selected atoms to be zero, which provides less energy reduction in each iteration of algorithm and reduces the efficiency.

II. Non-Negative OMP

Let $\mathbf{\Phi}_k = \mathbf{\Psi}_k \mathbf{R}_k$ be the QR factorisation of selected k atoms of dictionary $\mathbf{\Phi}$, where $\mathbf{\Psi}_k$ is a column orthonormal and $\mathbf{R}_k$ is an upper-triangle matrix. With some abuse of notation, we assume that in iteration k, $\mathbf{\Phi}_k$ is sorted based on the iteration number and $\boldsymbol{\phi}_i$, for $1 \leq i \leq k$, is the selected atom in ith iteration. In each iteration of NNOMP, we choose the atom which is positive and maximises $\mathbf{\Phi}^T\mathbf{y}$. In the first iteration, we do not need any orthogonalisation and we have $\boldsymbol{\phi}_1 = \boldsymbol{\psi}_1$ and $\mathbf{R} = [1]$. In the $1 \leq k$th iteration, let the best approximation of $\mathbf{y}$, using $\mathbf{\Phi}_k$, be $\sum_{i=1}^k \mathbf{x}_i\boldsymbol{\phi}_i = \sum_{i=1}^k \mathbf{z}_i\boldsymbol{\psi}_i$. In the $k+1$ iteration, we have,

$$\begin{aligned}\sum_{i=1}^{k+1} \mathbf{z}_i\boldsymbol{\psi}_i &= \sum_{i=1}^{k} \mathbf{z}_i\boldsymbol{\psi}_i + \mathbf{z}_{k+1}\boldsymbol{\psi}_{k+1} \\ &= \sum_{i=1}^{k} \mathbf{x}_i\boldsymbol{\phi}_i + \mathbf{z}_{k+1}\boldsymbol{\psi}_{k+1}\end{aligned}$$

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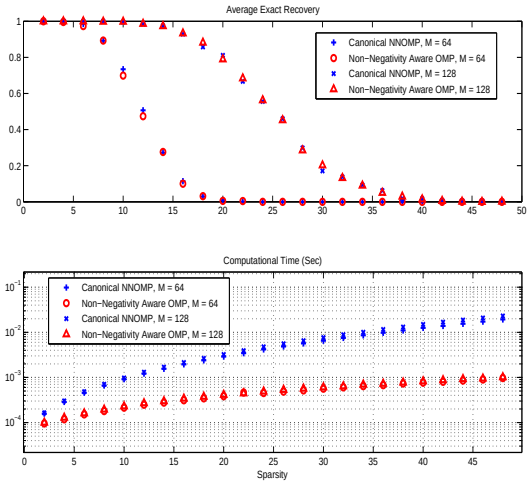

Fig. 1. Exact Recovery and Computation Time.

As $\boldsymbol{\psi}_{k+1}$ lives in the span of non-redundant set $\{\boldsymbol{\phi}_j\}_{j\in[1,k+1]}$, $\boldsymbol{\psi}_{k+1} = \sum_{j=1}^{k+1} \gamma_j\boldsymbol{\phi}_j$ for some unique γ_j's. We can then have,

$$\begin{aligned}\sum_{i=1}^{k+1} \mathbf{z}_i\boldsymbol{\psi}_i &= \sum_{i=1}^{k} \mathbf{x}_i\boldsymbol{\phi}_i + \sum_{j=1}^{k+1} \mathbf{z}_{k+1}\gamma_j\boldsymbol{\phi}_j \\ &= \sum_{i=1}^{k} (\mathbf{x}_i + \mathbf{z}_{k+1}\gamma_i)\boldsymbol{\phi}_i + \mathbf{z}_{k+1}\gamma_{k+1}\boldsymbol{\phi}_{k+1}.\end{aligned}$$

As $\mathbf{z}_{k+1}\gamma_{k+1}$ is always positive, we only need to assure that $\mathbf{x}_i + \mathbf{z}_{k+1}\gamma_i \geq 0$ or

$$\mathbf{z}_k \leq \min_{i,\gamma_i<0} \left|\frac{\mathbf{x}_i}{\gamma_i}\right|. \tag{1}$$

In the fast implementation of OMP using QR factorisation [2], we only need to update z_i at each iteration. To assure that $\mathbf{x}_i$'s are all non-negative, $\mathbf{z}_i$'s should comply the condition of (1). We then choose the atom that the corresponding $\mathbf{z}_{k+1}$, or shrunk by uper-bound of (1), has the largest value. It is worth mentioning that $\mathbf{R}_k$, $\mathbf{R}_k^{-1}$ and $\boldsymbol{\gamma}$ can be calculated very efficiently, using iterative reconstruction techniques.

III. Simulations

To demonstrate the performance, we randomly generated $\mathbf{\Phi}$ with 256 atoms and 64 or 128 rows using *i.i.d.* Gaussian noise, followed by column normalisation. With different sparsity and using proposed NNOMP and canonical NNOMP, we repeated the simulations 100 times. The exact recovery and computational time is shown in Figure 1. While the exact recovery is very similar, proposed method is significantly faster for large k's.

References

[1] A. Bruckstein, M. Elad, and M. Zibulevsky, "Sparse non-negative solution of a linear system of equations is unique," in *ISCCSP*, 2008, pp. 762–767.
[2] T. Blumensath and M. Davies, "On the difference between orthogonal matching pursuit and orthogonal least squares," Tech. Rep., 2007.