Constrained Overcomplete Analysis Operator Learning for Cosparse Signal Modelling

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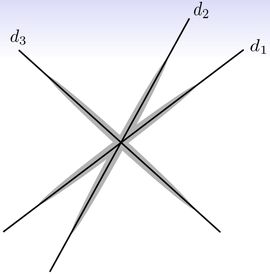
Outline:

Analysis Framework : An Introduction

- A low dimensional signal model.
- A special type of the union of subspaces signal model.
- Has many applications in, for example, denoising, compressed sensing and inverse problems to improve the overall performance.



Analysis Model

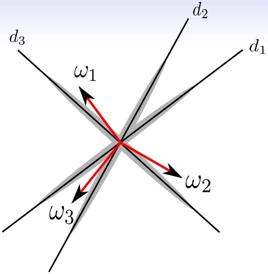
The signal $\mathbf{y}$ follows the model, if there exists a (linear) analysis operator $\Omega \in \mathbb{R}^{n \times m}$, $n \geq m$ that sparsifies $\mathbf{y}$,

$$\mathbf{z} = \Omega \mathbf{y}.$$

$\|\mathbf{z}\|_0 = n - p$, where $p > 0$ is called the co-sparsity of $\mathbf{y}$.

Analysis Operator Learning (AOL) Formulation

- A set of samples $\mathbf{Y} = [\mathbf{y}_1 \ \dots \ \mathbf{y}_i \ \dots \ \mathbf{y}_L]$ is given.
- The goal is to find a suitable analysis operator Ω such that $\|\Omega\mathbf{Y}\|_0$ is small.
- The objective is non-smooth $\Rightarrow$ not suitable for optimisation with variational techniques.
- A relaxation is to select the sum of absolute values operator, *i.e.* $\|\cdot\|_1 = \sum_{ij} |\{\cdot\}_{ij}|$.



Formulation

The learned operator can be found by minimising the sparsity promoting operator,

$$\min_{\Omega} \|\Omega\mathbf{Y}\|_1 \ \text{s.t.} \ \Omega \in \mathcal{C}$$

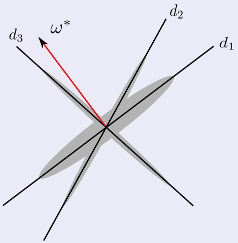
where $\mathcal{C}$ is a constraint, to exclude the trivial solutions, *e.g.* $\Omega = \mathbf{0}$.

Insufficient Constraints

Row norm constraints

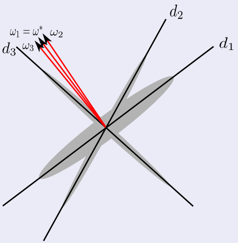
$$\forall i, \ \|\omega_i\|_2 = c$$

Rank one Ω_1 is found by repeating the best (almost) orthogonal direction ω^* to columns of $\mathbf{Y}$.



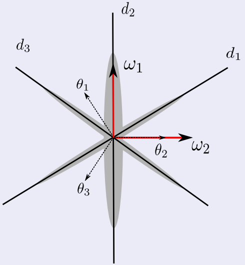
Row norm + full rank constraints

A randomly perturbed Ω from Ω_1 , *i.e.* row normalised $\Omega_1 + \mathbf{N}$, has a full rank and it is still not suitable.



Tight frame constraints

It resolves the issue in a complete setting. In the overcomplete cases, it includes zero-padded orthobases.



Proposed Constraint

Uniform Normalised Tight Frame (UNTF):

Definition: $\mathcal{C} = \{\Omega \in \mathbb{R}^{n \times m} : \Omega^T \Omega = \mathbf{I} \ \& \ \forall i \ \|\omega_i\|_2 = \sqrt{\frac{m}{n}}\}$

Pros and Cons:

- Zero-padded orthobases are not UNTF.
- There exist some practical methods to project onto the TF and the UN manifolds. However, there is no analytical way to find the projection onto the UNTF!
- There is no easy way to find the global optimum, using $\mathcal{C}$ as the constraint.

Projected Subgradient Algorithm for AOL

Motivation

Minimisation of a convex objective subject to the intersection of two manifolds $\Rightarrow$ a variant of projected subgradient algorithm is a good candidate.

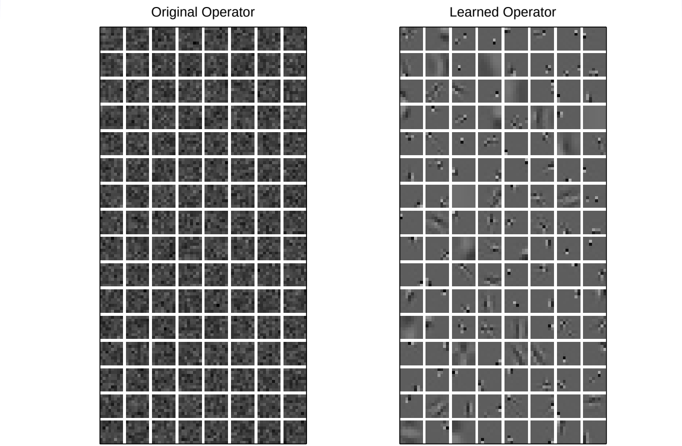
Projected Subgradient Type Algorithm for AOL

1: **initialisation:** $k = 1$, K_{max}, $\Omega^{[0]} = \mathbf{0}$, $\Omega^{[1]} = \Omega_{in}$, γ, $\epsilon \ll 1$
2: **while** $\epsilon \leq \|\Omega^{[k]} - \Omega^{[k-1]}\|_F$ and $k \leq K_{max}$ **do**
3: $\quad \Omega_G = \partial f(\Omega^{[k]})$
4: $\quad \Omega^{[k+1]} = \mathcal{P}_{UN}\left\{\mathcal{P}_{TF}\left\{\Omega^{[k]} - \gamma\Omega_G\right\}\right\}$
5: $\quad k = k + 1$
6: **end while**
7: **output:** $\Omega_{out} = \Omega^{[k-1]}$.

AOL for the Piecewise Constant Images

- Finding an Ω for the image patches of size 8×8.
- A 512×512 Shepp-Logan phantom image was used as the training image.
- $N = 16384$ image patches was randomly chosen from the training image.
- A pseudo-random UNTF operator $\Omega_0 \in \mathbb{R}^{128 \times 64}$ was used as the initial operator and the algorithm iterated 100,000 times!

AOL for the Piecewise Constant Images



Issues with the Projected Subgradinet Algorithm: Some Proposed Relaxations

- No analytical way to project onto UNTF $\rightarrow$ **no convergence** proof.
- Projection onto TF needs a full SVD calculation $\rightarrow$ **expensive implementation** and non-scalable algorithm.
- ℓ_1 term is not differentiable $\rightarrow$ **slow convergence** of the projected subgradient algorithm.

Relaxed AOL Formulation

1. **Relaxing the objective:** using a convex, but differentiable sparsity constraint $g(\Omega \mathbf{Y})$, where g is an entrywise function defined as,

$$g(x) = |x| - s \ln(1 + |x|/s), \; s \in \mathbb{R}^+$$

2. **Relaxing the constraint:** using quartic constraints $\|\Omega^T \Omega - \mathbf{I}\|_F^2 \leq \epsilon_{TF}$ and $\|\omega_i^T \omega_i - \frac{m}{n}\|_2^2 \leq \epsilon_{UN}, \quad \forall i \in [1, n]$

Relaxed Analysis Operator Learning

Relaxed Analysis Operator Learing Formulation

An unconstrained objective is generate by using two Lagrange multipliers γ and λ:

$$f(\mathbf{\Omega}) = g(\mathbf{\Omega Y}) + \frac{\gamma}{4}\|\mathbf{\Omega}^T\mathbf{\Omega} - \mathbf{I}\|_F^2 + \frac{\lambda}{4}\sum_i \left\{\|\omega_i^T\omega_i - \frac{m}{n}\|_2^2\right\}.$$

- f is differentiable and it is also convex, if we restrict its domain to $\mathcal{C}_c = \{\mathbf{\Omega} : \mathbf{\Omega}^T\mathbf{\Omega} - \mathbf{I} \succeq \mathbf{0},\ \forall i,\ (\omega_i^T\omega_i - \frac{m}{n}) \geq 0\}$.

Gradient Descent Algorithm for AOL

A variable step-size gradient descent, with line search, can be used to minimise $f(\mathbf{\Omega})$, where the gradient of f can easily be found by:

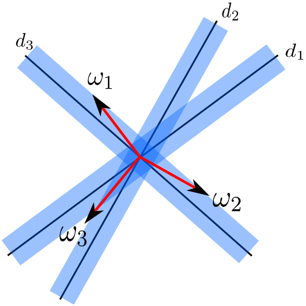
$$\nabla f = \left[\frac{\mathbf{Z}_{i,j}}{s + |\mathbf{Z}_{i,j}|}\right]_{i,j}\mathbf{Y}^T + \gamma\left(\mathbf{\Omega\Omega}^T - \mathbf{I}\right)\mathbf{\Omega} + \lambda\left[\omega_i\left(\omega_i^T\omega_i - \frac{m}{n}\right)\right]_i^T$$

$$\mathbf{Z} := \mathbf{\Omega Y}$$

Noise Aware Analysis Operator Learning

Approximately Cosparse Exemplars

- Training data $\mathbf{Y}$ is approximately cosparse, $\mathbf{Y} = \mathbf{Y}_c + \mathbf{N}$, where $\mathbf{N}$ is **noise** or **model mismatch** and $\mathbf{Y}_c$ is cosparse.
- The goal is to find an operator $\mathbf{\Omega}$, such that $\mathbf{\Omega Y}_c$ has many zeros.
- The issue is that **we do not know $\mathbf{Y}_c$ precisely!** A solution is to somehow approximate it.
- This is indeed very similar to the **dictionary learning** problem, where we do not know the sparse coefficients.



Noise Aware Analysis Operator Learning: Formulation and Algorithm

Noise Aware Analysis Operator Learning

$$\min_{\mathbf{\Omega},\widehat{\mathbf{Y}}} \|\mathbf{\Omega}\widehat{\mathbf{Y}}\|_1 + \frac{\theta}{2}\|\widehat{\mathbf{Y}} - \mathbf{Y}\|_F^2 \quad \text{s.t.} \quad \mathbf{\Omega} \in \mathcal{C}.$$

Solving by alternating minimisation technique.

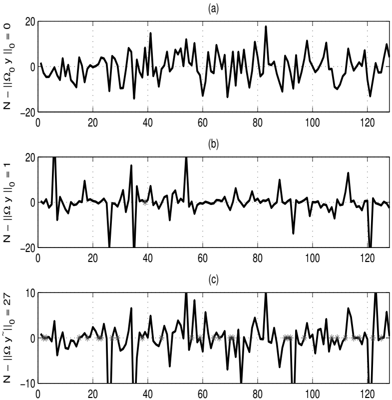
- Optimisation based on $\mathbf{\Omega}$: similar to noise-less AOL.
- Optimisation based on $\widehat{\mathbf{Y}}$: a convex program. $\rightarrow$ Douglas-Rachford Splitting (DRS) technique was used to efficiently solve the program.
- Algorithm usually converges after a few number of alternating minimisations.
- For the optimisation base on $\widehat{\mathbf{Y}}$, the ℓ_1 penalty can be relaxed, similar to the operator update step, and the new convex program can be solved using a gradient descent algorithm with a line search.

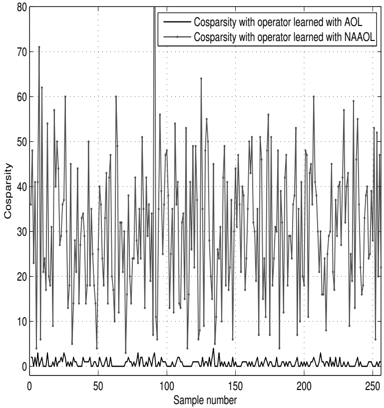
An Operator for the Face Images: Setting

- Learning an Ω for the image face patches from the Yale face database.
- $L = 16384$, 8×8 image patches were randomly selected from different faces.

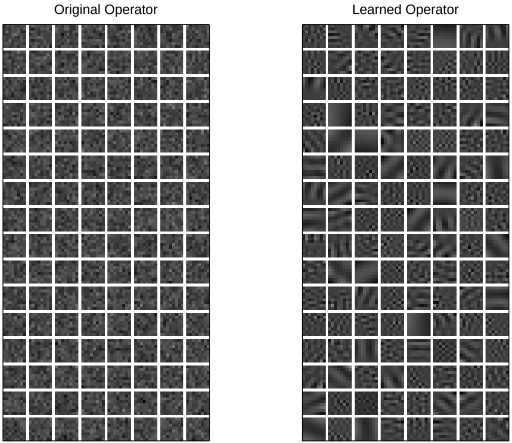
An Operator for Face Images: Cosparsity Comparison

- The analysis coefficients $\mathbf{z} = \mathbf{\Omega y}$ and cosparsities were calculated, using $\mathbf{\Omega}_0$, $\mathbf{\Omega}_{AOL}$ and $\mathbf{\Omega}_{NAAOL}$.



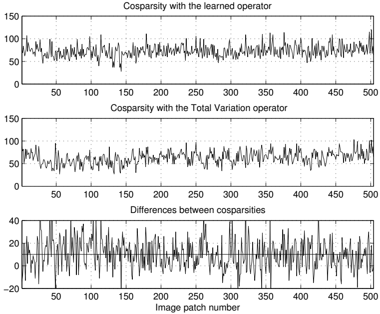


Learned Operator



Face Images Denoising: TV v.s. Learned Operator

- TV operator for comparison.
- Two different regularisation parameters, $\lambda = 0.3$ & 0.1 .



Conclusion and Future Work

Conclusion:

- The constrained analysis operator learning is a useful technique to find a suitable analysis operator.
- The proposed constraint can be relaxed to reduce the complexity of the optimisation algorithm, while including some **approximately UNTF** operators.
- The simulation results emphasis on the fact that we should use the correct analysis operator, i.e. TV or oscillatory operators.
- The convergence of the relaxed AOL is guaranteed, as its objective has a bounded curvature and its sublevel set is conpact.

Future Work:

- Investigating the **local identifiability** of operators in this framework.
- More investigations on the **structures** of the learned operators.

Thanks for your attention.