Cosparsifying Overcomplete Analysis Operator Learning

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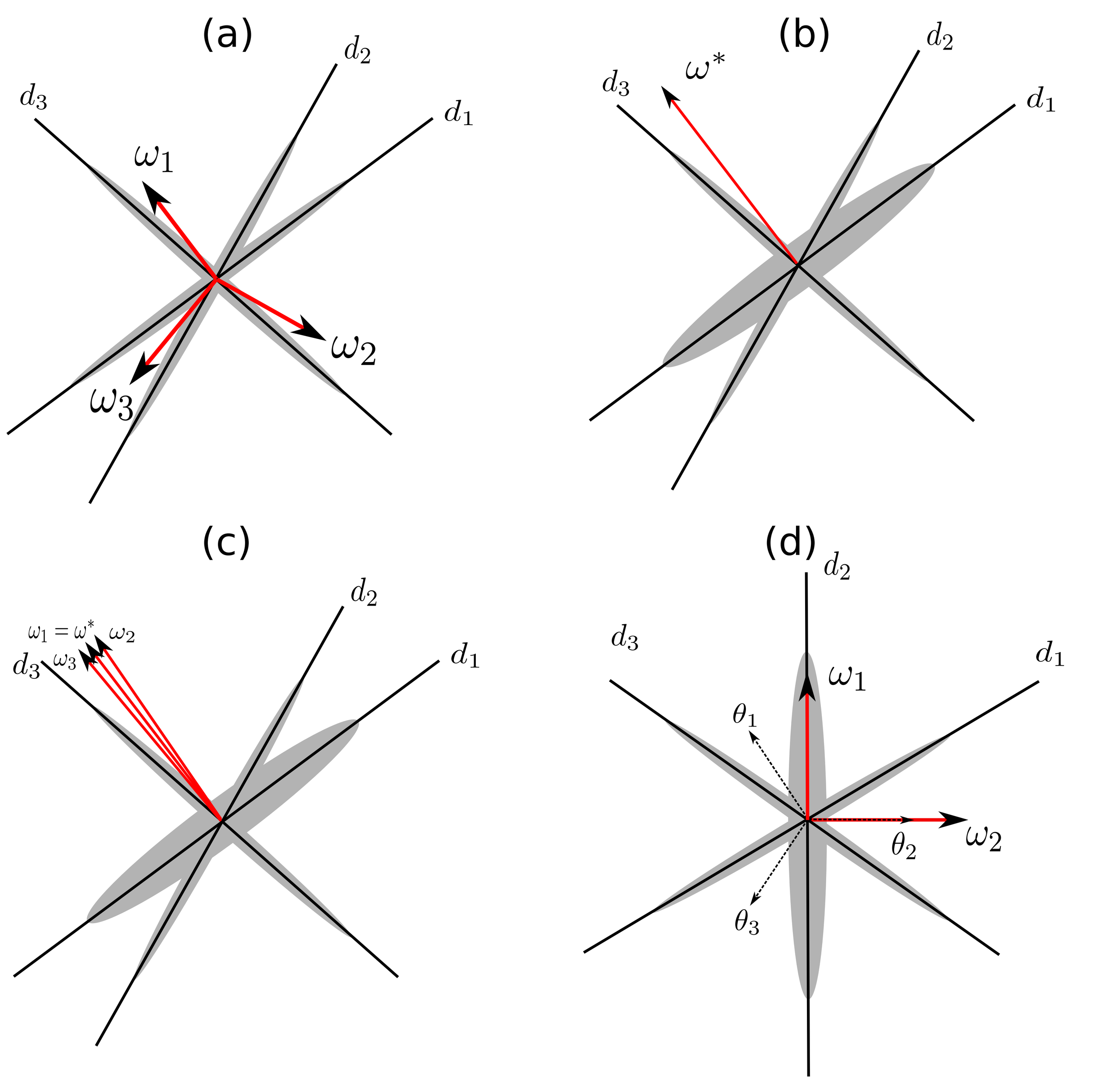
Abstract — *We consider the problem of learning a low-dimensional signal model from a collection of training samples. The mainstream approach would be to learn an overcomplete dictionary to provide good approximations of the training samples using sparse synthesis coefficients. This famous sparse model has a less well known counterpart, in analysis form, called the cosparse analysis model. In this new model, signals are characterised by their parsimony in a transformed domain using an overcomplete (linear) analysis operator. We propose to learn an analysis operator from a training corpus using a constrained optimisation framework based on ℓ_1 minimisation. The reason for introducing a constraint in the optimisation framework is to exclude trivial solutions. Although there is no final answer here for which constraint is the most relevant constraint, we investigate some conventional constraints in the model adaptation field and use the uniformly normalised tight frame (UNTF) for this purpose. We then derive a practical learning algorithm, based on projected subgradients and Douglas-Rachford splitting technique, and demonstrate its ability to robustly recover a ground truth analysis operator, when provided with a clean training set, of sufficient size. We also find an analysis operator for images, using some noisy cosparse signals.*



Constrained Analysis Operators Learning Formulation

The aim of analysis operator learning is to find an operator $\mathbf{\Omega} \in \mathbb{R}^{a\times n}$, adapted to a set of observations of the signals $\mathbf{Y} = [\mathbf{y}_i] \in \mathbb{R}^{n\times l}, \mathbf{y}_i = x_i + n_i$, where $\mathbf{\Omega Y}$ is sparse. In this setting, $\mathbf{Y}$ is called (*approximately*) cosparse. With ℓ_1 sparsity measure, a formulation for finding $\mathbf{\Omega}$ is as follows,

$$\min_{\mathbf{\Omega},\mathbf{X}} \|\mathbf{\Omega X}\|_1 + \frac{\lambda}{2}\|\mathbf{Y}-\mathbf{X}\|_F^2, \ \text{s.t.} \ \mathbf{\Omega} \in \mathcal{C}$$

where $\mathcal{C}$ is a constraint. We need $\mathcal{C}$, to avoid trivial solutions, *e.g.* $\mathbf{\Omega} = \mathbf{0}$.

- Row norm constraints: the optimum solution is obtained by repeating the best row ω^* , i.e., $\mathbf{\Omega}_1^* := [\omega_i = \omega^*]^T_{i\in[1,a]}$.
- Row norm + full rank constraints: the optimum solutions have very small condition numbers, *e.g.* $\mathcal{P}_{\mathcal{C}_F}\{\epsilon\mathbf{A} + \mathbf{\Omega}_1^*\}$, where $\mathcal{P}_{\mathcal{C}_F}$, $\mathbf{A}$ and ϵ respectively are row normalisation, a random Gaussian matrix and a very small constant.
- Tight frame (TF) constraint: the optimum solutions are the zero-padded bases.
- Proposed constraint: Uniform Normalised Tight Frame (UNTF):

$$\mathcal{C} = \{\mathbf{\Omega} \in \mathbb{R}^{a\times n} \ : \ \mathbf{\Omega}^T\mathbf{\Omega} = \mathbf{I}, \ \forall i \ \|\omega_i\|_2 = \sqrt{\frac{a}{n}}\}.$$

- Pros and Cons:

1. Zero-padded orthobases are not UNTF.
2. Efficient methods exist to project onto the TF and the uniform normalised (UN) manifolds. However, there is no analytical way to find the projection onto the UNTF!
3. There is no easy way to find the global optimum, using $\mathcal{C}$ as the constraint.

Alternating Minimisation AOL Algorithm

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initialisation: $\mathbf{\Omega}^{[0]}$, $\mathbf{X}^{[0]} = \mathbf{Y}$, $i = 0$,
while not converged **do**
  $\mathbf{\Omega}^{[i+1]} = \text{argmin}_{\mathbf{\Omega}\in\mathcal{C}} \|\mathbf{\Omega X}^{[i]}\|_1$,
  $\mathbf{X}^{[i+1]} = \text{argmin}_{\mathbf{X}} \|\mathbf{\Omega}^{[i+1]}\mathbf{X}\|_1 + \frac{\lambda}{2}\|\mathbf{Y}-\mathbf{X}\|_F^2$
  $i = i + 1$
end while.

Analysis Operator Update

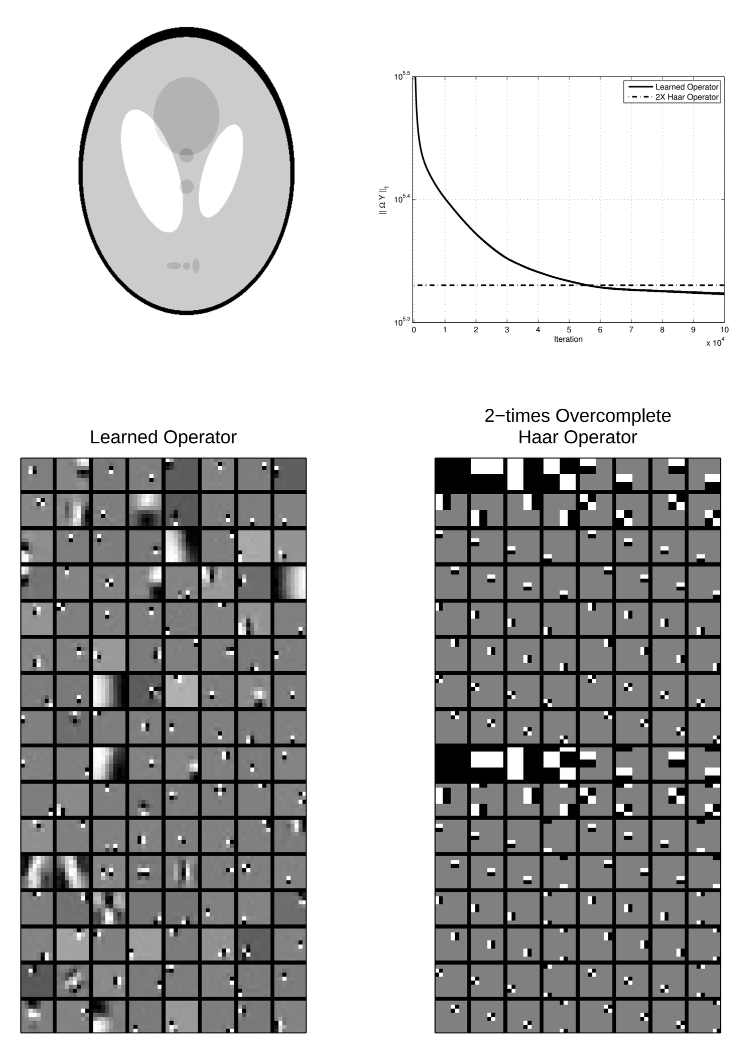
initialization: $k = 1$, K_{max}, $\mathbf{\Omega}^{[0]} = \mathbf{0}$, $\Omega^{[1]} = \mathbf{\Omega}_{in}$, γ, $\epsilon \ll 1$
while $\epsilon \leq \|\mathbf{\Omega}^{[k]} - \mathbf{\Omega}^{[k-1]}\|_F$ and $k \leq K_{max}$ **do**
  $\mathbf{\Omega}_G = \partial f(\mathbf{\Omega}^{[k]})$
  $\mathbf{\Omega}^{[k+1]} = \mathcal{P}_{UN}\left\{\mathcal{P}_{TF}\left\{\mathbf{\Omega}^{[k]} - \gamma\mathbf{\Omega}_G\right\}\right\}$
  $k = k + 1$
end while
output: $\mathbf{\Omega}_{out} = \mathbf{\Omega}^{[k-1]}$.

Cosparse Signal Update

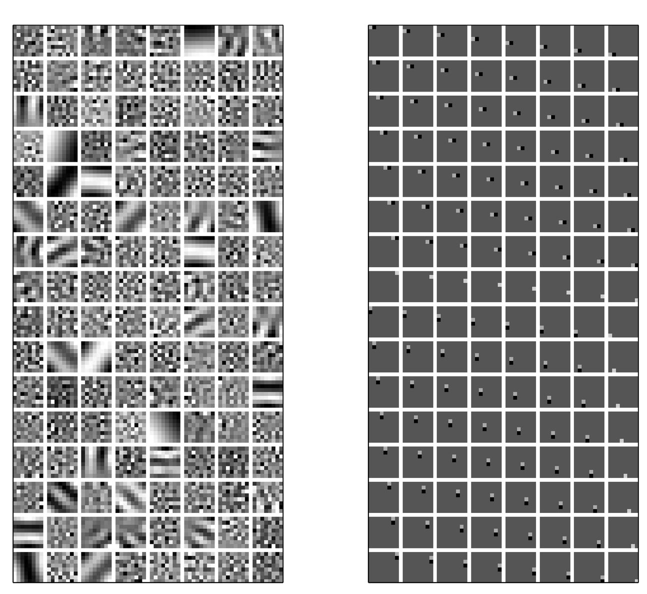
initialisation: $k = 1$, $\mathbf{X}_{[k]} = \mathbf{X}^{[i]}$, $\mathbf{\Omega} = \mathbf{\Omega}^{[i+1]}$, $\mathbf{B}_{[k]} = \mathbf{0}$, $\mathbf{Z}_{[k]} = \mathbf{\Omega X}_{[k]}$
while $\epsilon \leq \|\mathbf{X}_{[k]} - \mathbf{X}_{[k-1]}\|_F$ and $k \leq K_{\max}$ **do**
  $\mathbf{X}_{[k+1]} = \frac{1}{\lambda+\gamma}(\lambda\mathbf{Y} + \gamma\mathbf{\Omega}^T(\mathbf{Z}_{[k]} - \mathbf{B}_{[k]}))$
  $\mathbf{Z}_{[k+1]} = \mathcal{S}_{\frac{1}{\gamma}}\left\{\mathbf{\Omega X}_{[k+1]} + \mathbf{B}_{[k]}\right\}$
  $\mathbf{B}_{[k+1]} = \mathbf{B}^{[k]} + (\mathbf{\Omega X}_{[k+1]} - \mathbf{Z}_{[k+1]})$
  $k = k + 1$
end while
output: $\mathbf{X}^{[i+1]} = \mathbf{X}_{[k-1]}$

Simulations and Summary

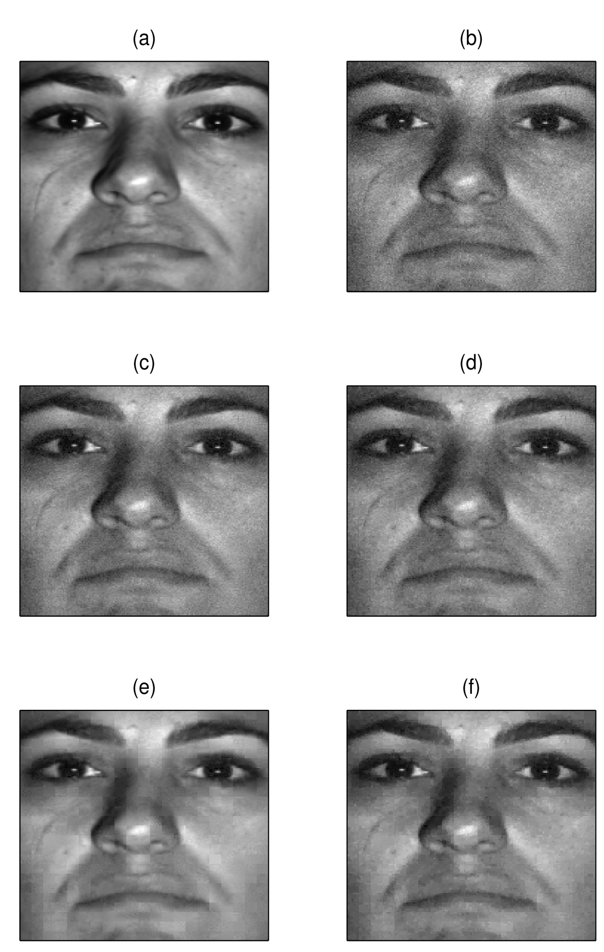
Noiseless AOL for Image Patches



Noise Aware AOL for Face Image Patches



Average PSNR (dB)	$\lambda = 0.3$	$\lambda = 0.1$
Finite Difference	28.9643	31.5775
Learned Operators	29.0254	32.5202



Summary

- The proposed analysis operator learning technique showed promising results in finding a suitable operator for images.
- Although the proposed constraint may not be the most relevant constraint, it works very well in practice.
- Each iteration of the Analysis Operator Update needs a full singular value decomposition, which is an issue in the scalability of algorithm.
- The parameter splitting is a very efficient technique for the optimisations in the analysis framework.

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[1] M. Yaghoobi, S. Nam, R. Gribonval and M. E. Davies, *"Constrained Overcomplete Analysis Operator Learning for Cosparse Signal Modelling"*, submitted, `http://arxiv.org/abs/1205.4133`.